[7] Loc. cit., reference 1, pp. 106–107.

[8] H. A. Schwartz, "Gesammelte math. Abhandlungen," **1**, pp. 111, note (7).

[9] L. Bieberbach, "Über die konforme Kreisabbildung nahezu kreisförmiger Bereiche," *Sitzungsber. preuss. Akadie. Wiss., Math.-Phys. Klasse*, 1924, pp. 181–188.

[10] F. Riesz, "Über die Randwerte einer analytischen Funktion," *Math. Zeitschrift*, **8**, 1923, p. 95.

[11] Cf. Kerékjártó, "Vorlesungen über Topologie," passim.

[12] I found this theorem in trying to verify the statement of Schwartz referred to in § 3; I developed its proof, in a more general form, in a talk in the math. colloquium in Leipzig, June, 1925.

[13] T. Radó, *Aufgabe*, **41**, and H. Kneser, *Lösung der Aufgabe*, **41**, *Jahresbericht der deutschen Mathematiker-Vereinigung*, **35**, 1926.

THE PROBLEM OF THE CALCULUS OF VARIATIONS IN m-SPACE WITH END-POINTS VARIABLE ON TWO MANIFOLDS

BY ELDRED CURRIER

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

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The results which we shall state in this paper have been proved in detail and will be published at an early date.

We are concerned with a calculus of variations problem in the ordinary parametric form, with an integrand which is positive, of class C^4, and homogeneous in the usual way.[1] The domain of the coördinates is a closed region R in m-space.

Let g be an extremal segment in R, and let M_r and M_s be two regular manifolds of class C^3 and of dimensionalities r and s, respectively, $(0 \leqq r, s \leqq m - 1)$ which cut g transversally at the distinct points A and B, respectively. Suppose that the integrand F is positively regular along g.

Let K be the class of all regular curves of class D', whose end-points lie on M_r and M_s, respectively. The classical problem of the calculus of variations is to determine necessary and sufficient conditions that the integral take on a proper relative minimum along g from M_r to M_s with respect to its values on curves of the class K. In this paper we present a solution of the problem.

Let H_r be the family of extremals in the neighborhood of g cut transversally by M_r. Let the equations of H_r be given in terms of regular parameters[2] $(t, v_1, \ldots, v_n)$ so chosen that $(v) = (v^\circ)$ gives g. We prove that the equations of H_r can be given in this form with regular parameters.[2] Let $a_{ij}(t)$, $(i, j = 1, \ldots, m)$ be the elements of the jacobian of H_r; with respect to the parameters (t, v), evaluated at $(v) = (v^\circ)$. Points P on g at which the determinant of the elements $a_{ij}(t)$ vanishes are said to be focal points of M_r, and the order of the vanishing of the determinant is said to be the order of the corresponding focal point. It is proved that the point A is a focal point of M_r of order $n - r$, $n = m - 1$.

Let $t = a$ and $t = b$ $(a < b)$ be the values of the parameter t on g at the points A and B, respectively. Let the focal points of M_r on g exclusive of those on $a < t < b$ be the points: $t = d_1, d_2$, etc., and the points $t = c_1$ c_2, etc., where

$$\ldots c_3 \leqq c_2 \leqq c_1 \leqq a < b \leqq d_1 \leqq d_2 \leqq d_3 \ldots, \text{ etc.} \qquad (1)$$

where focal points are counted according to order so that the equality sign holds between k successive terms of (1) corresponding to a focal point of order k.

Similarly, let the focal points of M_s except those on $a < t < b$ be:

$$\ldots c_3' \leqq c_2' \leqq c_1' \leqq a < b \leqq d_1' \leqq d_2' \leqq d_3' \ldots, \text{ etc.} \qquad (2)$$

THEOREM 1. *In order that the integral take on a relative minimum along g it is necessary that there be no focal points of M_r or of M_s on $a < t < b$, and that:*

$$\ldots c_2' \leqq c_2,\ c_1' \leqq c_1,\ d_1' \leqq d_1,\ d_2' \leqq d_2 \ldots, \text{ etc.} \qquad (3)$$

Let the Weirstrass E-function[3] $E(z, r, \sigma)$ be positive for (z, r) in the neighborhood of $(z, \dot{z})$ on g, and (σ) any set not (o) nor proportional to (r). This will be called the Weirstrass condition.

THEOREM 2. *Suppose that the integrand F is positively regular along g, and that the Weirstrass condition is satisfied. In order that the integral take on a proper relative minimum along g it is sufficient that there be no focal points of M_r or of M_s on $a < t < b$, and that at least one of the following conditions be satisfied:*

$$\ldots,\ c_2' < c_{n+1},\ c_1' < c_n,\ d_n' < d_1,\ d_{n+1}' < d_2,\ \ldots, \text{ etc.} \qquad (4)$$

The necessary and sufficient conditions of theorems one and two, respectively, do not coincide very closely. In the special case of the problem of the calculus of variations in the plane, conditions (4) are the same as conditions (3), except for the equality signs. This is because in the plane $n = 1$. In this special case, the conditions have been obtained by Bliss.[4]

We have obtained essentially all the conditions as indicated by the following two theorems.

THEOREM 3. *Assume A and B are not conjugate. Let Σ denote any condition on the relative distribution of focal points which is not a consequence of the necessary conditions of Theorem 1. There exists a pair of manifolds M and M′ cutting g transversally at A and B, respectively, whose focal points have a distribution which fails to satisfy the condition Σ, and yet for which g gives a weak minimum.*

THEOREM 4. *Let there be no pairs of conjugate points on g and let the Weirstrass and regularity conditions be satisfied. Let Σ be any condition on the relative distribution of the points $d_1, \ldots, d_n$ and $d_1', \ldots, d_n'$ which is not a consequence of the sufficient conditions (4) of Theorem 2. There exists a pair of manifolds M and M' cutting g transversally at A and B, respectively, whose focal points satisfy the condition Σ, and which have no focal points on $a < t < b$, and yet for which g fails to give even a weak minimum.*

We have obtained other necessary and sufficient conditions which coincide more closely than the conditions of theorems one and two.

Let H_s be the field of extremals in the neighborhood of g cut transversally by M_s, and let $b_{ij}(\gamma)$, $(i, j = 1, \ldots, m)$ be the elements of the jacobian of H_s with respect to regular parameters $(\gamma, u_1, \ldots, u_n)$ evaluated at $(u) = (u^\circ)$, where (u°) is the set of parameters of g in the field H_s. Let P be a point on g which is not a focal point of M_r or of M_s. Let the parameters (t, v) and (γ, u) be chosen so that at the point P on g:

$$a_{ij} = b_{ij},\ (i, j = 1, \ldots, m). \tag{5}$$

THEOREM 5. *The positive and negative type numbers and the nullity of the form $D(v)$, where:*

$$D(v) = a_{ik} F_{r_i r_j} (\dot{a}_{jh} - \dot{b}_{jh}) v_h v_k,\ (i, j = 1, \ldots, m;\ k, h = 2, \ldots, m) \tag{6}$$

are numerical invariants of the point P and the manifolds M_r and M_s with respect to admissible[1] space transformations and with respect to such admissible parameter transformations[2] as preserve the relationship (5).

The arguments of the partial derivatives of F in (6), are $(z, \dot{z})$ on g at P; and a_{ik}, $\dot{a}_{jh}$, $\dot{b}_{jh}$ are taken at P.

THEOREM 6. *A necessary condition that the integral take on a relative minimum along g is that the form $D(v)$ be positive indefinite, and that there be no focal points of M_r or of M_s on $a < t < b$.*

THEOREM 7. *Granting that F is positively regular on g and the Weirstrass condition is satisfied, a sufficient condition that the integral take on a proper relative minimum along g is that there be no focal points of M_r or of M_s on $a < t < b$, and the form $D(v)$ be positive definite.*

Thus the problem of the minimum has been solved, but there remains the general problem of classifying extremal segments g cut transversally by M_r and M_s according to the negative type number and the nullity of a fundamental quadratic form.

We cut across the segment of g between M_r and M_s by p successive regular n-manifolds t_i of class C^3 not tangent to g and so close together that no pairs of conjugate points occur on the closed segments of g between successive manifolds. Let R and S be points on M_r and M_s, respectively, and let P_i be a point on t_i. If the points $(R, P_1, \ldots, P_P, S)$ are close to

g they can be joined by unique successive extremal segments forming a broken extremal E. Let (u) be the set of $\mu = r + s + pn$ parameters of which the first r are regular parameters of the point R on M_r, the next n are regular parameters of the point P_1 on t_1, etc., until finally the last s are regular parameters of the point S on M_s. The value of the integral taken along the broken extremal E will be a function of the variables (u) and will be denoted by $J(u)$. The function $J(u)$ will have a critical point when $(u) = (u^\circ)$, where (u°) are the parameters of g in the family of broken extremals E. We define the form Q as

$$Q = J_{ij}u_iu_j, \quad (i, j = 1, \ldots, \mu) \tag{7}$$

where the arguments of the partial derivatives of J are (u°). We classify the extremal segments cut transversally by M_r and M_s according to the negative type number and the nullity of the form Q.

Let P be any point on the open segment of g between M_r and M_s which is not a focal point of M_r or of M_s. Let q_r be the sum of the orders of the focal points of M_r on the open segment of g between M_r and P, and let q_s be the sum of the orders of the focal points of M_s on the open segment of g between P and M_s. Let $D(v)$ be the invariant form (6) constructed at P.

THEOREM 8. *The nullity of the form Q is equal to the nullity of the form $D(v)$.*

THEOREM 9. *The negative type number of the form Q is equal to*

$$q_r + N + q_s \tag{8}$$

where N is the negative type number of the form $D(v)$.

[1] Marston Morse, "The Foundations of the Calculus of Variations in the Large in m-Space" (first paper), *Trans. Amer. Math. Soc.*, **31** (1929), pp. 380–381.

[2] The parameters (t, v) of H_r will be said to be *regular* if the equations are of class C^2, and if (v) = const. gives the members of H_r as regular 1-manifolds in terms of the parameter t, and if the determinant of $a_{ij}(t)$ vanishes, if at all, at isolated points. A parameter transformation which carries (t, v) into another set of regular parameters is said to be *admissible.*

[3] Bliss, "The Weirstrass E-Function for Problems of the Calculus of Variations in Space," *Trans. Amer. Math. Soc.*, **15** (1914), pp. 369–378.

[4] Bliss, "Jacobi's Criterion When Both End Points Are Variable," *Math. Ann.*, **58** (1903), p. 70.